

Why Noether symmetry of $F(R)$ theory yields three-half power law?

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Abstract

Noether symmetry of $F(R)$ theory of gravity reveals $F(R) = R^{\frac{3}{2}}$, if the expression for the scalar curvature for R-W metric is treated as a constraint and entered into the action through a Lagrange multiplier. In the process, a cyclic coordinate is found which gives a solution that appears to explain the present cosmological evolution. The interesting issue is that out of infinite curvature invariant terms, Noether symmetry selects only $R^{\frac{3}{2}}$. Here, we explore the very speciality and study the attractive features and shortcomings of such term in the context of cosmological evolution.

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1 Introduction

In the last decade several path finder models have been explored which enunciated the best fit with SNIa data set. For most of these models SNIa data set implies a smooth transition from a decelerated phase to a recent accelerated phase of cosmic evolution, and the presence of nearly 74% of dark energy, in disguise. However, starting from the Λ CDM concordance model, often cited as the standard model, all the models suffer from some sort of disease. The order of magnitudes of the vacuum energy density (corresponding to cosmological constant Λ) calculated by the Quantum field theorists and the Cosmologists differ by a huge (10^{120}) amount. Quintessence model solves the problem, but it requires a scalar field with a suitable potential. But then, we need a scalar with some exotic potential, which is not at hand presently. Further, Quintessence model does not suffice to handle the situation with state parameter $\omega < -1$, which is not excluded in view of the presently available data. It requires additional scalar field containing kinetic term with reverse sign. Thus, the situation gets more and more complicated. There was attempt to solve the puzzle assuming particle creation phenomena at the expense of gravitational field [1], [2], which has strong theoretical basis, but there is no direct experimental evidence as yet. The only thing that is left is curvature, which can not be detected directly. Inflation in the early Universe may be an artefact of R^2 term in gravity [3] and comic acceleration in the late Universe might have been caused by R^{-1} term [4]. Thus, it appears that an action in the form $S = \int [\alpha R + \beta R^2 + \gamma R^{-1}] \sqrt{-g} d^4x$, having pure gravitational origin, might be the one to solve the cosmic puzzle from early Universe till date.

However, R^{-1} term fails to produce Newtonian gravity in the weak energy limit and does not comply with the solar test [5]. Even worse is, it shows unavoidable instabilities within matter in the weak gravity limit [6] and fails to explain big bang nucleosynthesis (BBN) [7]. Further, for any R^{-n} term, with $n > 0$, it has been shown [8] that the scale factor during the matter phase grows as $t^{\frac{1}{2}}$, instead of the standard $t^{\frac{2}{3}}$ - law, prior to the transition to an accelerated expansion. This is grossly inconsistent with cosmological observations viz., WMAP

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(Wilkinson Microwave Anisotropy Probe) and LSS (Large Scale Structure). Therefore one requires to find some scalar invariant other than R^{-1} , suitable to explain late time cosmic acceleration and that may pass the solar test. Such a theory, dubbed as $F(R)$ theory of gravity [see [9] and references therein] is much more attractive, promising and intriguing than scalar field models. But then, there are at least a couple of associated problems. Firstly, out of indefinitely many curvature invariant terms, one requires an elegant method to choose suitable ones that fit all cosmological data at hand. Next, it is practically impossible to solve the field equations of $F(R)$ theory of gravity, corresponding to any nonlinear form of R . Both the problems presumably may be solved by invoking Noether symmetry.

In order to express higher order theory of gravity in non-degenerate and canonical form, it is required to be spanned by an auxiliary variable in addition to the basic variables. In the process, the configuration space of the Lagrangian gets enlarged. Thus, one can demand the existence of Noether symmetry amongst the field variables (viz., between the scale factor and the auxiliary variable) even when the theory is reduced to a measure zero subset of its configuration space, i.e., if the Robertson-Walker minisuperspace model is accounted for. This was attempted largely by several authors [10], [11], to find a suitable form of $F(R)$, spanning the Lagrangian by a set of configuration space variables (a, \dot{a}, R, \dot{R}) . Noether symmetry of $F(R)$ theory of gravity [10], [11], yields $F(R) = R^{\frac{3}{2}}$, in the Robertson-Walker minisuperspace model, which appears to be important in studying early Universe but not the late. However, explicit solution indicates a smooth transition from early deceleration to the late time cosmic acceleration [11], which is definitely an attractive feature of such a curvature invariant term. Nevertheless, the more interesting issue is that, out of infinite number of curvature invariant terms, Noether symmetry selects $R^{\frac{3}{2}}$ term in particular. The result is doubtful for the following reason. In a series of works [12], [13] it has been pointed out that although classical field equations can be formulated choosing any auxiliary variable, different auxiliary variables produce different Hamiltonian. As a result, different quantum descriptions are revealed. A toy model [13] explains the reason, which is, different auxiliary variables produce different definitions of canonical momenta. In the classical field equations, it does not make any difference, since one requires derivative of momenta. But Hamiltonian depends on momenta itself, and is different for different choice of canonical momenta. In finding Noether symmetry, the situation is the same. Here again we require momenta, and not its derivative and so different auxiliary variable might produce different results for Noether symmetry. The standard technique of finding auxiliary variable is to take partial derivative of the action with respect to the variable with highest derivative appearing in the action, i.e., $Q = \frac{\partial S}{\partial \dot{h}_{ij}}$ for the Robertson-Walker line element, where h_{ij} is the metric on the three space. Thus, we think that the study of Noether symmetry of $F(R)$ theory of gravity requires further attention.

Despite the fact that $R^{\frac{3}{2}}$ theory of gravity has some attractive features, particularly in explaining the late time cosmic evolution it suffers from the same type of problems associated with R^{-n} , with $n > 0$ theories of gravity, since it never ends up with Newtonian analogue in the weak field limit. This is also apparent from the solution obtained [11]. The solution depicts that in the matter dominated era, prior to the transition to an accelerated expansion, the scale factor never follows the standard $t^{\frac{2}{3}}$ -law. Such disease may be cured if such an action would have been supplemented by Einstein-Hilbert action. Hence, in the following section we try to explore Noether symmetry of $F(R)$ gravity being supplemented by the Einstein-Hilbert term. This is done to understand if the technique of finding such symmetry is correct. If it is, it is then supposed to yield the same result, since $\alpha R + \beta F(R) = \gamma F_1(R)$, and this is what we have obtained. For a further check, in section 3, we explore Noether symmetry of $R^{\frac{3}{2}}$ term following the standard technique and end up with the same solutions obtained earlier[11]. Thus we understand that Noether symmetry of pure gravity, yields nothing more than $F(R) = R^{\frac{3}{2}}$. This initiates to find what is so special in such a curvature invariant term. In section 4, under a change of variable h_{ij} , we found that Noether symmetry is in-built in $R^{\frac{3}{2}}$ term, since $h_{ij} = a^2$ becomes cyclic and the field equations are solved at ease yielding the same known solution. This implies that one does not require to request [11] Noether symmetry of $R^{\frac{3}{2}}$ curvature invariant term. In section 5, we review Noether symmetry of $F(R)$ theory of gravity, in view of the changed variable h_{ij} , and observe that the conserved current may be solved quite easily and it is not required to find cyclic variable any more. In the next section 6, we introduce matter and found solutions both in the radiation and in the matter dominated era. Section 7 is devoted to explore how far $R^{\frac{3}{2}}$ term is acceptable in the context of available cosmological data. In section 8, we conclude with our findings.

2 In search of Noether symmetry for Einstein-Hilbert action being modified by $F(R)$ theory of gravity.

We have already discussed the problem of $F(R)$ theory of gravity if it does not contain Einstein-Hilbert part in the action. Noether symmetry of $F(R)$ theory of gravity yields $F(R) = R^{\frac{3}{2}}$. The solution is encouraging as far as late time cosmic acceleration is concerned. But the problem is that, it only passes transiently through $a \propto t^{\frac{2}{3}}$ in the matter era before the transition to accelerated phase [11], and thus fails to explain WMAP and LSS, as in the case of R^{-n} term, [7], [8]. So, we try to find Noether symmetry for the $F(R)$ theory being coupled to the Einstein-Hilbert sector. Although it appears to be trivial, since these two terms together may be combined to get yet another $F(R)$ theory, however, it is important to see if it is true while Noether symmetry is being explored. In the process, it turns out to be an important test to check if the technique of exploring Noether symmetry is correct. Hence we start with the following action,

$$A = \int \sqrt{-g} d^4x \left[\frac{R}{16\pi G} + BF(R) \right] - \frac{1}{8\pi G} \int [\sqrt{h}K] d^3x - 2B \int [\sqrt{h}F_{,R}K] d^3x, \quad (1)$$

where, $\frac{1}{8\pi G} \int [\sqrt{h}K] d^3x$ is the Gibbons-Hawking boundary term and $\Sigma = 2B \int [\sqrt{h}F_{,R}K] d^3x$, is the surface term for $F(R)$ theory of gravity that emerges under variational principle (metric formalism). In the above, h is the determinant of the metric on three space, K is the trace of the extrinsic curvature and $F_{,R}$ is the derivative of $F(R)$ with respect to R . Note that this surface term Σ reduces to the Gibbons-Hawking boundary term taking $F(R) = \frac{1}{16\pi G}R$ and that for the curvature squared action $F(R) = R^2$ [12]. Action (1) in the background of isotropic and homogeneous Robertson-Walker line element,

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \quad (2)$$

leads to

$$A = \int \left[\frac{6}{16\pi G} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) + BF(R) - \lambda \left\{ R - 6 \left(\frac{\ddot{a}}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right\} \right] a^3 dt d^3x - \frac{1}{8\pi G} \int [\sqrt{h}K] d^3x - 2B \int [\sqrt{h}F_{,R}K] d^3x \quad (3)$$

where, the expression for R has been introduced as a constraint through Lagrange multiplier λ . Now varying the above action with respect to R , one gets $\lambda = BF_{,R}$, which is substituted in the action. Now under integration by parts the surface terms get cancelled and the action may be expressed as, (taking $\frac{3}{8\pi G} = M$),

$$A = \int \left[-Ma\dot{a}^2 + Mka + Ba^3(F - RF_{,R}) - 6Ba\dot{a}^2F_{,R} - 6Ba^2\dot{a}F_{,RR}\dot{R} + 6BkaF_{,R} \right] dt, \quad (4)$$

and the Noether equation may be written as,

$$\begin{aligned} XL = \mathcal{L}_X L = & \alpha(-M\dot{a}^2 + Mk + 3Ba^2(F - RF_{,R}) - 6B\dot{a}^2F_{,R} - 12Ba\dot{a}\dot{R}F_{,RR} + 6BkF_{,R}) \\ & + \beta(-Ba^3RF_{,RR} - 6Ba\dot{a}^2F_{,RR} - 6Ba^2\dot{a}\dot{R}F_{,RRR} + 6BkaF_{,RR}) \\ & + \left(\frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial R} \dot{R} \right) (-2Ma\dot{a} - 12Ba\dot{a}F_{,R} - 6Ba^2\dot{R}F_{,RR}) + \left(\frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial R} \dot{R} \right) (-6Ba^2\dot{a}F_{,RR}) = 0, \end{aligned} \quad (5)$$

where, X is the vector field. Now we equate the coefficients of \dot{a}^2 , \dot{R}^2 , $\dot{a}\dot{R}$ and others respectively to zero and obtain,

$$(M + 6BF_{,R})(\alpha + 2a\alpha_{,a}) + 6B(\beta + a\beta_{,a})aF_{,RR} = 0. \quad (6)$$

$$6Ba^2F_{,RR}\alpha_{,R}=0. \quad (7)$$

$$6B\beta a^2F_{,RRR}+6B(2\alpha+a\alpha_{,a}+a\beta_{,R})aF_{,RR}+2(M+6BF_{,R})a\alpha_{,R}=0. \quad (8)$$

$$\alpha[Mk+3B(F-RF_{,R})a^2+6BkF_{,R}]+B\beta[6k-a^2R]aF_{,RR}=0. \quad (9)$$

In the above equations comma (,) represents derivative. Equation (7) implies $\alpha \neq \alpha(R)$, since, we require nonlinear form of $F(R)$, ie., $F_{,RR} \neq 0$. So, we have, $\alpha = \alpha(a)$, and separating $\beta = A(a)S(R)$. Equation (8) is expressed as,

$$S\frac{F_{,RRR}}{F_{,RR}}+\frac{dS}{dR}=-\frac{1}{A}\left[2\frac{\alpha}{a}+\frac{d\alpha}{da}\right]=c_1, \quad (10)$$

where, c_1 is a separation constant. The above equation (10) yields,

$$\frac{d\alpha}{da}+2\frac{\alpha}{a}=-c_1A, \quad (11)$$

and,

$$\frac{F_{,RRR}}{F_{,RR}}+\frac{S_{,R}}{S}=\frac{c_1}{S}. \quad (12)$$

Equation (6) can as well be separated using the separation constant c_2 as,

$$S=\frac{\frac{M}{6B}+F_{,R}}{c_2F_{,RR}}, \quad (13)$$

and

$$a^2A_{,a}+aA+c_2(2a\alpha_{,a}+\alpha)=0. \quad (14)$$

Equations (11) and (14) are combined to give the following differential equation,

$$a^2\alpha_{,aa}+(3-2c_1c_2)a\alpha_{,a}-c_1c_2\alpha=0 \quad (15)$$

In view of equations (13) and (15), equation (9) takes the following form,

$$k(M+6BF_{,R})[(c_1c_2-2)\alpha-a\alpha_{,a}]+3Bc_1c_2(F-RF_{,R})\alpha a^2+\frac{(M+6BF_{,R})}{6}(a\alpha_{,a}+2\alpha)a^2R=0. \quad (16)$$

Thus, one can solve α in view of equation (15) and hence A from (11). $F(R)$ and $S(R)$ may be solved in view of equations (12) and (13). Finally, equation (16) is satisfied for $c_1c_2=2$. At the end, following set of solutions exists for $k=\pm 1, 0$.

$$\alpha=\frac{n}{a}, \quad \beta=-\frac{2nR}{a^2}, \quad \mathcal{F}=\frac{1}{\sqrt{R}}(a\dot{R}-2\dot{a}R)=\frac{a^3}{\sqrt{R}}\frac{d}{dt}\left(\frac{R}{a^2}\right), \quad (17)$$

where, n is a constant of integration and \mathcal{F} is the conserved current. However, the most interesting result is,

$$F(R)=-\frac{M}{6B}R+CR^{\frac{3}{2}}, \quad (18)$$

which, under the substitution $M = \frac{3}{8\pi G}$, reduces the action to

$$A = \int \sqrt{-g} d^4x [DR^{\frac{3}{2}}]. \quad (19)$$

Thus, Noether symmetry of $F(R)$ does not admit an explicit Einstein-Hilbert term in the $F(R)$ theory of gravity. This is the result that we may expect if the technique of exploring Noether symmetry, treating the expression of R as a constraint is correct. So, in the following we further check, what Noether symmetry yields, starting from $R^{\frac{3}{2}}$ term, instead of $F(R)$.

3 Noether symmetry of $R^{\frac{3}{2}}$ - standard technique.

This section is important for the following reason. To express $F(R)$ theory of gravity in non-degenerate and canonical form, one usually introduces the form of R as a constraint through a Lagrange multiplier λ . The extended configuration space in such case is (a, \dot{a}, R, \dot{R}) . In the standard formalism, where a form of curvature invariant term is fixed, the configuration space is different viz., (a, \dot{a}, Q, \dot{Q}) . Here, Q is an auxiliary variable chosen as the derivative of the action with respect to the highest derivative of the variable appearing in the action, that has not been possible to integrate by parts to produce a surface term. We here check, if Noether symmetry along with the solutions presented by [11] exists, when action containing $R^{\frac{3}{2}}$ term is spanned by such a set of configuration space variables, i.e., whether Noether symmetry is an artifact of a particular choice of configuration space variables or not. One may expect that since Noether symmetry of $F(R)$ theory yields $R^{\frac{3}{2}}$, so $\mathcal{L}_X L = 0$ will be exactly solved if $F(R) = R^{\frac{3}{2}}$ is assumed a-priori. However, we show below that this is not the situation. The action under consideration in the Robertson-Walker minisuperspace model is,

$$S = B \int R^{\frac{3}{2}} \sqrt{-g} d^4x - 2B \int [\sqrt{h} F_{,RK}] d^3x = B \int 6^{\frac{3}{2}} [a\ddot{a} + \dot{a}^2 + k]^{\frac{3}{2}} dt - 9\sqrt{6}B[a\ddot{a} + \dot{a}^2 + k]^{\frac{1}{2}} a\dot{a}. \quad (20)$$

Now let us define the auxiliary variable Q as,

$$Q = \frac{\partial S}{\partial \dot{a}} = 9\sqrt{6}B[a\ddot{a} + \dot{a}^2 + k]^{\frac{1}{2}} a. \quad (21)$$

Expressing the action in non-degenerate, canonical form,

$$S = \int \left[\frac{Q}{a} [a\ddot{a} + \dot{a}^2 + k] - \frac{1}{1458B^2} \frac{Q^3}{a^3} \right] dt - 9\sqrt{6}B[a\ddot{a} + \dot{a}^2 + k]^{\frac{1}{2}} a\dot{a}, \quad (22)$$

and integrating by parts, we obtain,

$$S = \int \left[-\dot{Q}\dot{a} + \frac{Q}{a}(\dot{a}^2 + k) - \frac{1}{1458B^2} \frac{Q^3}{a^3} \right] dt + Q\dot{a} - 9\sqrt{6}B[a\ddot{a} + \dot{a}^2 + k]^{\frac{1}{2}} a\dot{a}, \quad (23)$$

One can easily check that the last two terms get cancelled and so, there is no controversy on the choice of the auxiliary variable. The field equations are,

$$\ddot{Q} - 2\dot{Q}\frac{\dot{a}}{a} - 2Q\frac{\ddot{a}}{a} + Q\frac{\dot{a}^2}{a^2} - Q\frac{k}{a^2} + \frac{1}{486B^2} \frac{Q^3}{a^4} = 0. \quad (24)$$

$$-\dot{a}\dot{Q} + Q\frac{\dot{a}^2}{a} - Q\frac{k}{a} + \frac{1}{1458B^2} \frac{Q^3}{a^3} = 0. \quad (25)$$

Equation (24) is a fourth order equation in the scale factor (a), while equation (25) is third order and apparently it is impossible to solve the pair. So, it is better to ask Noether symmetry between the field variables a and Q , i.e., $\mathcal{L}_X L = XL = 0$, which reads,

$$\alpha \left[\frac{Q^3}{486B^2a^4} - \frac{\dot{a}^2 + k}{a^2} Q \right] + \beta \left[\frac{\dot{a}^2 + k}{a} - \frac{Q^2}{486B^2a^3} \right] + (\alpha_a \dot{a} + \alpha_Q \dot{Q}) \left[2\frac{\dot{a}}{a} Q - \dot{Q} \right] - (\beta_a \dot{a} + \beta_Q \dot{Q}) \dot{a} = 0. \quad (26)$$

Equating coefficients of \dot{a}^2 , \dot{Q}^2 , $\dot{a}\dot{Q}$ and terms free from derivative, we get following four equations, which are linear in α and β , viz.,

$$\alpha Q - 2a\alpha_{,a}Q - a\beta + a^2\beta_{,a} = 0. \quad (27)$$

$$\alpha_{,Q} = 0. \quad (28)$$

$$\alpha_{,a} - 2\frac{Q}{a}\alpha_{,Q} + \beta_{,Q} = 0. \quad (29)$$

$$(\alpha Q - \beta a)(486B^2ka^2 - Q^2) = 0. \quad (30)$$

In the above comma represents derivative, as before. Equation (28) dictates that $\alpha \neq \alpha(Q)$, while equation (30) clearly yields two cases, viz.,

$$\beta = \frac{\alpha}{a}Q, \quad \text{and}, \quad Q = 9\sqrt{6}B\sqrt{k}a.$$

3.1 Case - I, $\beta = \frac{\alpha}{a}Q$

Equation (29) in view of the above condition yields,,

$$\alpha = \frac{\alpha_0}{a}, \quad \text{and so,} \quad \beta = \frac{\alpha_0}{a^2}Q. \quad (31)$$

It is quite easy to verify that equation (27) is trivially satisfied for the above solutions. Thus the conserved current \mathcal{F} is,

$$\mathcal{F} = \alpha p_a + \beta p_Q \implies \frac{d}{dt} \left(\frac{Q}{a} \right) = \text{Constant}. \quad (32)$$

Plugging in the definition of Q from relation (21), one can immediately solve the above equation for a as,

$$a = \left[a_4 t^4 + a_3 t^3 + \left(\frac{3a_3^2}{8a_4} - k \right) t^2 + a_2 t + a_1 \right]^{\frac{1}{2}}, \quad (33)$$

where, a_1 , a_2 , a_3 and a_4 are integration constants. It is not difficult to check that the above solution satisfies both the field equations (24) and (25). This is exactly the solution obtained and analyzed by Cappelletti et al [11]. Note that, the solution has been found directly from Noether symmetry as we found earlier in the case of R^2 theory of gravity being coupled with a scalar field [14] and Gauss-Bonnet dilatonic interaction [16] and we don't require to solve the field equations any more. This is a unique feature of higher order theory of gravity. The above vacuum solution may be treated as one corresponding to the early Universe, when curvature played the dominant role. The Universe started evolving as $a \propto \sqrt{t}$, and soon transits to a power law inflationary era.

3.2 Case - II, $Q = 9\sqrt{6k}Ba$

Since Q is proportional to a , so, the field variables practically reduce to one, and it is nonsense to try to find Noether symmetry. Rather, we can check, if such a relation between Q and a yields a solution in view of the definition of Q given in (21) that satisfies the field equations (24) and (25). The solution is,

$$a = \sqrt{d_1 t + d_0}, \quad (34)$$

where, d_1 and d_0 are integration constants. This solution satisfies the field equation (24), but equation (25) is satisfied only under the condition, $k = 0$. This is trivial, since then $Q = 0$ and the field equations are trivially

satisfied. In view of the definition of Q , the Ricci scalar $R = 0$, and so, it automatically leads to Friedmann solution in the radiation dominated era in the following manner. The combination of Friedmann equations,

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G\rho \quad \text{and} \quad 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho,$$

together yield,

$$6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = R = 8\pi G(\rho - 3p).$$

Now, in the radiation dominated era, $\rho - 3p = 0$, and so the above equation reduces to,

$$6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = R = 0.$$

Further if $k = 0$, the solution of the above equation is $a \propto t^{\frac{1}{2}}$.

4 What is so special in $R^{\frac{3}{2}}$? A different look with change of variable.

So, we observe that the technique of exploring Noether symmetry for $F(R)$ theory of gravity treating the expression of R as a constraint of the theory appears to be alright, since standard technique does not yield anything else. Hence, the question that naturally arises, what is so special in the curvature invariant term $R^{\frac{3}{2}}$. In this section, we pose to answer this question. It has already been emphasized that the basic variable is h_{ij} and not the scale factor a . In the case of higher order theory of gravity, the configuration space gets enlarged and the new variables that finally come into rescue is K_{ij} [13]. So, here we start with the basic variable, $h_{ij} = a^2 = z$. In view of the expression of Ricci scalar,

$$R = 6\left(\frac{\ddot{z}}{2z} + \frac{k}{z}\right),$$

The action now reads,

$$S = B \int 3\sqrt{3}(\ddot{z} + 2k)^{\frac{3}{2}} dt - 2B \int [\sqrt{h}F_{,R}K] d^3x. \quad (35)$$

The auxiliary variable is

$$Q = \frac{\partial S}{\partial \ddot{z}} = \frac{9\sqrt{3}}{2}B(\ddot{z} + 2k)^{\frac{1}{2}}, \quad (36)$$

and the canonical form of the action is

$$S = \int \left[Q(\ddot{z} + 2k) - \frac{4Q^3}{729B^2} \right] dt - 2B \int [\sqrt{h}F_{,R}K] d^3x = \int \left[-\dot{Q}\dot{z} + 2kQ - \frac{4Q^3}{729B^2} \right] dt. \quad (37)$$

The definition of the auxiliary variable is restored immediately from the Q variation equation, while the variable z turns out to be cyclic. Thus the field equations are oversimplified to,

$$\ddot{Q} = 0, \quad (38)$$

$$\dot{z}\dot{Q} + 2kQ - \frac{4Q^3}{729B^2} = 0, \quad (39)$$

In view of the definition of the auxiliary variable Q given in (36), equation (38) immediately gives the following solution,

$$a = \left[a_4 t^4 + a_3 t^3 + \left(\frac{3a_3^2}{8a_4} - k \right) t^2 + a_2 t + a_1^2 \right]^{\frac{1}{2}}, \quad (40)$$

which resembles exactly with the solution (33) obtained earlier in view of Noether symmetry. Note that, equation (39) also yields the same above solution. Thus, it is clear that Noether symmetry is in built in $R^{\frac{3}{2}}$ theory of gravity, since z turns out to be cyclic automatically.

5 Revisiting Noether symmetry of $F(R)$ theory of gravity

We now understand that to find Noether symmetry for $F(R)$ theory of gravity, it is better to choose the variable, $h_{ij} = a^2 = z$, rather than the scale factor a . In this section we show that under such a choice of variable, Noether symmetry of $F(R)$ theory of gravity yields a rather handy conserved current, which can be solved at ease, and it is not required to search for the cyclic co-ordinate.

$$A = B \int \left[F(R) - \lambda \left\{ R - 6 \left(\frac{\ddot{z}}{2z} + \frac{k}{z} \right) \right\} \right] z^{\frac{3}{2}} dt - 2B \int \left[\sqrt{h} F_{,R} K \right] d^3x. \quad (41)$$

Following the same technique, i.e., varying the action with respect to R and setting it to zero, one finds $\lambda = F_{,R}$, which when substituted in (41) and the surface term is accounted for, through integration by parts, the action may be expressed in the following non-degenerate and canonical form,

$$A = B \int \left[(F - RF_{,R}) z^{\frac{3}{2}} - \frac{3}{2} F_{,R} \frac{\dot{z}^2}{\sqrt{z}} - 3F_{,RR} \sqrt{z} \dot{R} \dot{z} + 6k F_{,RR} \sqrt{z} \right] dt. \quad (42)$$

Demanding Noether symmetry one obtains,

$$\begin{aligned} & \alpha \left[\frac{3}{2} (F - RF_{,R}) \sqrt{z} + F_{,R} \frac{3\dot{z}^2}{4z^{\frac{3}{2}}} - F_{,RR} \frac{3\dot{R}\dot{z}}{2\sqrt{z}} + 3k \frac{F_{,R}}{\sqrt{z}} \right] - \beta \left[F_{,RR} \left(Rz^{\frac{3}{2}} + \frac{3\dot{z}^2}{2\sqrt{z}} - 6k\sqrt{z} \right) + 3F_{,RRR} \sqrt{z} \dot{R} \dot{z} \right] \\ & - (\alpha_{,z} \dot{z} + \alpha_{,R} \dot{R}) \left(3F_{,R} \frac{\dot{z}}{\sqrt{z}} + 3F_{,RR} \sqrt{z} \dot{R} \right) - (\beta_{,z} \dot{z} + \beta_{,R} \dot{R}) (3F_{,RR} \sqrt{z} \dot{z}) = 0. \end{aligned} \quad (43)$$

Equating coefficients as usual, following four equations emerge,

$$\frac{\alpha}{2z} - 2\alpha_{,z} = \frac{F_{,RR}}{F_{,R}} (\beta + 2z\beta_{,z}). \quad (44)$$

$$F_{,RR} \sqrt{z} \alpha_{,R} = 0. \quad (45)$$

$$\beta \frac{F_{,RRR}}{F_{,RR}} + \beta_{,R} + \frac{\alpha}{2z} + \alpha_{,z} + \frac{\alpha_{,R} F_{,R}}{z F_{,RR}} = 0. \quad (46)$$

$$\alpha \left[\frac{3}{2} (F - RF_{,R}) \sqrt{z} + 3k \frac{F_{,R}}{\sqrt{z}} \right] = \beta \left[F_{,RR} \left(Rz^{\frac{3}{2}} - 6k\sqrt{z} \right) \right]. \quad (47)$$

For a non linear form of $F(R)$, i.e., $F_{,RR} \neq 0$, equation (45) implies that $\alpha = \alpha(z)$ only. Thus, equation (44) demands separation of variable β as, $\beta = \beta_1(z)\beta_2(R)$, and is expressed as,

$$\frac{\frac{\alpha}{2z} - 2\alpha_{,z}}{\beta_1 + 2z\beta_{1,z}} = \beta_2 \frac{F_{,RR}}{F_{,R}} = c_1, \quad (48)$$

while, equation (46) is expressed as,

$$-\frac{\frac{\alpha}{2z} + \alpha_{,z}}{\beta_1} = \beta_2 \frac{F_{,RRR}}{F_{,RR}} + \beta_{2,R} = c_2, \quad (49)$$

where, c_1 and c_2 are separation constants. It is not difficult to check that middle terms of the above two equations are the same, i.e., $\beta_2 \frac{F_{,RR}}{F_{,R}} = \beta_2 \frac{F_{,RRR}}{F_{,RR}} + \beta_{2,R}$ and so, $c_1 = c_2 = c$. Thus equations (48) and (49) may be solved and one gets,

$$\alpha = mz + n \quad \text{and} \quad \beta_1 = -\frac{1}{2cz}(3mz + n). \quad (50)$$

In view of the solutions (50), equation (47), after simplification takes the following form,

$$3(mz + n)F - 12kmF_{,R} - 2nRF_{,R}, \quad (51)$$

which is satisfied only under the condition $m = 0$. Hence one can solve equation (51) for $F(R)$ and then equation (48) for $\beta_2(R)$. The final set of solutions thus obtained is,

$$\alpha = n, \quad \beta = -\frac{nR}{z}, \quad F(R) = R^{\frac{3}{2}}, \quad (52)$$

where, n is a constant. The conserved current now reads,

$$\mathcal{F} = \frac{9n}{2} \frac{d}{dt}(\sqrt{zR}), \quad (53)$$

which is having the same form as obtained earlier by Capozziello-et al [11]. Substituting the expression for R , the above conserved current can now be solved directly and one obtains,

$$a = \left[a_4 t^4 + a_3 t^3 + \left(\frac{3a_3^2}{8a_4} - k \right) t^2 + a_2 t + a_1 \right]^{\frac{1}{2}}, \quad (54)$$

which is the same solution obtained earlier in (33) and (40).

6 Presence of matter

The solution (33), or ((40) or (54)), that we have presented so far corresponds to vacuum, which may have some importance in the early curvature dominated Universe, since it represents transition from decelerating Universe with $a \propto \sqrt{t}$ to a power law inflation. To unveil the importance of $R^{\frac{3}{2}}$ in the late stage of cosmic evolution, one has to take some form of matter into account. So, let us start again from the action,

$$S = \int \left[BR^{\frac{3}{2}} - Ma^{-3(w+1)} \right] \sqrt{g} d^4x - 2B \int \left[\frac{3}{2} \sqrt{hR} K \right] d^3x, \quad (55)$$

where, $w = \frac{p}{\rho}$ is the state parameter. Note that, in the radiation dominated era, $w = \frac{1}{3}$, and so, the second term reads, $Ma^{-3(w+1)} = \frac{\rho_{r0}}{a^4}$ and in the matter era, $w = 0$, and it reads $Ma^{-3(w+1)} = \frac{\rho_{m0}}{a^3}$. Here, ρ_{r0} , and ρ_{m0} refer to radiation density and the matter density at the present epoch, which has together been represented by M . Following the same prescription of introducing auxiliary variable given in (36), and after taking care of the surface term, the above action can be expressed in the following canonical and non-degenerate form,

$$S = \int \left[-\dot{Q}\dot{z} + 2kQ - \frac{4}{729B^2}Q^3 - Mz^{-\frac{3w}{2}} \right] dt. \quad (56)$$

The field equations are,

$$\ddot{Q} + \frac{3Mw}{2}z^{-(\frac{3w+2}{2})} = 0, \quad (57)$$

$$\dot{Q}\dot{z} + 2kQ - \frac{4}{729B^2}Q^3 = Mz^{-\frac{3w}{2}}, \quad (58)$$

In the radiation dominated era $w = \frac{1}{3}$, and the field equations admit a solution in the form,

$$\sqrt{z} = a = b_0 t^{\frac{3}{4}} + b_1, \quad (59)$$

under the condition, $k = 0$ and $B = -(\frac{32M}{135})$, instead of the standard Friedmann type, i.e., $a \propto t^{\frac{1}{2}}$. This creates problem in explaining BBN, since the Universe expands at a much faster rate than the standard Friedmann model. Further, due to such a high expansion rate in the radiation era, it becomes difficult for the seed of perturbations to accumulate matter, in the matter dominated era, to form structures. Now, in the matter dominated era $w = 0$, and so is \dot{Q} . Thus we recover the same solution (33) or, ((40) or (54)), i.e.,

$$\sqrt{z} = a = \left[a_4 t^4 + a_3 t^3 + \left(\frac{3a_3^2}{8a_4} - k \right) t^2 + a_2 t + a_1 \right]^{\frac{1}{2}}, \quad (60)$$

which is true for arbitrary curvature parameter, viz., $k = 0, \pm 1$.

7 Fitting the observational data

Data fitting of the above unique solution for $R^{\frac{3}{2}}$ in the matter dominated era, has been presented already by Capozziello-et al [11]. Setting $H_0 t_0 = 1$ and deceleration parameter $q_0 = -0.04$, they reduced the solution to a one parameter model, for which a nice fit with Λ CDM is obtained. However, for such parametric values, the redshift at decoupling, $z_{dec} = 1100$, occurs at $t = 11 \times 10^6$ years, which is clearly in contradiction with the WMAP data and the standard model. Further, the Hubble parameter $H(z)$ versus redshift z curve (figure 1), is largely different from the standard model at high redshift. For the required fit, they [11] also took some finite value of k , viz., $k = -0.49$. Here we observe that the only power law solution that satisfies the field equation in the radiation dominated era, requires $k = 0$. So, we are required to set $k = 0$, in the matter dominated era also. In the solution above, one can set $a_1 = 1$, without any loss of generality. Still, we are left with three parameters, viz., a_2 , a_3 and a_4 . We have studied following two cases.

7.1 $H_0 t_0 = 1$.

Fixing up $H_0 t_0 = 1$, one can now express, one of the parameters of the theory (we have chosen a_2) in terms of the other two (viz., a_3 and a_4 here). The important observation is that the fit with SNIa data is almost perfect for a truly wide range of parametric values of a_3 and a_4 . We briefly present our result in table 1. The table depicts that the present value of the state parameter w_{e0} , remains close to -1 for a huge range of parametric values of $10^{-3} < a_4 < 10^6$ ($a_4 \neq 0$) and $10^{-5} < a_3 < 10^6$. Within such a range of values the transition redshift z_a always remains close to 1. The Redshift-Distance modulus curve fits almost perfectly with the Λ CDM model and thus the SNIa-data in disguise, taking the density parameter of the baryonic and nonbaryonic dark matter $\Omega_m = 0.26$ and the rest, $\Omega_\Lambda = 0.74$, as the density parameter of the dark energy. One of such fits has been shown in figure 2. Despite such amazingly attractive result, some serious problems have also been encountered. Almost in all the cases the redshift value is $z \approx 200$ at around $t = 3.75 \times 10^{-4}$ Billion years. Note that according to the standard model it is the time after Big-Bang, when photons decoupled and whose redshift value is $z_{dec} \approx 1100$, as confirmed by WMAP data. This is practically a huge problem. Further, considerably large deviation from the standard model in the early matter dominated epoch is clearly visible from figure 3, where a plot of the Hubble parameter $H(z)$ versus the redshift parameter z is drawn with the parametric values $a_4 = 1, a_3 = 0.1$. The two, deviates even further at a large redshift, and the feature is the same for all other parametric values. Such a deviation from the standard model in the early matter dominated era is inconsistent with the WMAP data and

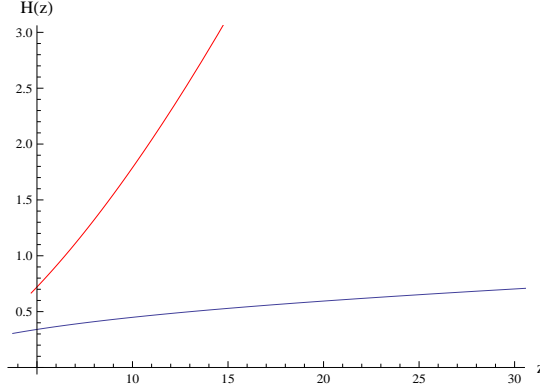


Figure 1: Hubble parameter $H(z)$ versus the Redshift parameter z for the present model (blue, lower curve) shows a huge deviation at large redshift, with the standard model $a \propto t^{\frac{2}{3}}$ (red) for the parameters chosen by Capozziello-et al [11].

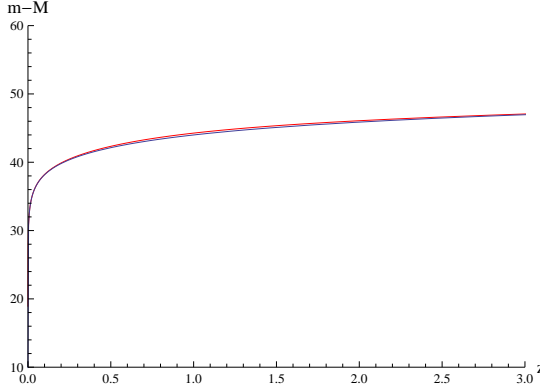


Figure 2: Distance modulus ($M - m$) versus redshift z plot of the present model (blue), shows perfect fit with the Λ CDM model (red) taking $a_4 = 1$ and $a_3 = 0.1$. The same feature is observed almost for all parametric values of a_3 and a_4 .

LSS [8]. Finally, Figure 4 is the effective state parameter w_e versus redshift z plot for the same above parametric values. It shows that the effective state parameter w_e remains close to 0.33 at large z , which clearly depicts that the early matter dominated era is far from the standard Friedmann model ($w_e = 0$) and tracks $a \propto t^{\frac{2}{3}}$ only transiently.

7.2 $H_0 t_0 = 1$ and $z_{dec} = 1100$, for $t = 3.75 \times 10^5$ years.

In the preceding section we have noticed that the unique solution obtained for $R^{\frac{3}{2}}$ action in the matter dominated era shows excellent fit with Λ CDM model and hence SNIa data in disguise, for a huge range of parametric values, but it fails to fit WMAP data corresponding to the early matter dominated epoch. Hence, here we fix up the WMAP data a-priori, which fixes up yet another parameter of the theory, viz., a_3 , in the present case. Thus, we end up with a one-parameter model. Here, we observe that again for a wide range of parameter, $a_4 > 10^{-5}$, with $a_4 \neq 0$, the Redshift-Distance modulus curve fit is almost perfect, like the one presented in figure 2. The present value of the effective state parameter is $w_{e0} \approx -0.35$ in almost all the cases, which is nowhere near Λ CDM model. But then, w_{e0} is model dependent, and so it creates no problem as such. However, the transition redshift is unacceptably large, the minimum of which is $z_a = 4.2$. Further, the Hubble parameter $H(z)$ versus redshift z plot (figure 5), here again shows huge deviation which increases rapidly with z . The only difference is the Hubble parameter for the present model here, remains far below than that corresponding to the standard model. The effective state parameter again tracks 0.33, instead of $w_e = 0$, in the early matter dominated epoch, which is grossly inconsistent with the LSS.

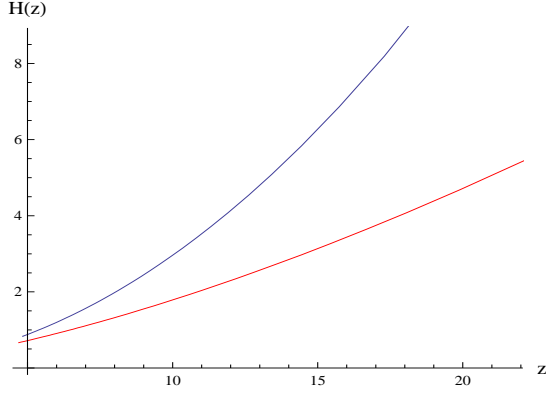


Figure 3: Hubble parameter $H(z)$ versus the Redshift parameter z for the present model (blue) shows a wide deviation at large redshift, with the standard model $a \propto t^{\frac{2}{3}}$ (red) for $H_0 t_0 = 1, h = 0.72, a_4 = 1$ and $a_3 = 0.1$. Blue being on the top.

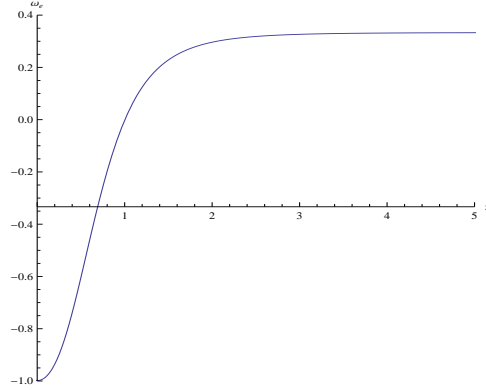


Figure 4: The effective state parameter w_e versus versus the Redshift parameter z for the present model ($a_4 = 1, a_3 = 0.1$) shows that in the early epoch of matter domination the state parameter tracks the value $w_e = 0.33$, which means the model always tracks $a \propto t^{\frac{1}{2}}$ rather than the standard model $a \propto t^{\frac{2}{3}}$. This creates problem to explain large scale structure.

Table 1: Data fitting with $H_0 t_0 = 1$ and $h = 0.72$

a_4	a_3	w_{e0}	z_a
10^5	10^4	-0.998	0.72
10^5	10^5	-0.984	0.70
10^5	10^6	-0.866	0.68
10	0	-1	1
1	0.1	-1	0.68
1	0.001	-0.999	1
5	1	-0.997	1
0.01	0.00079	-1	1
10^{-3}	7.78×10^{-5}	-1	0.71

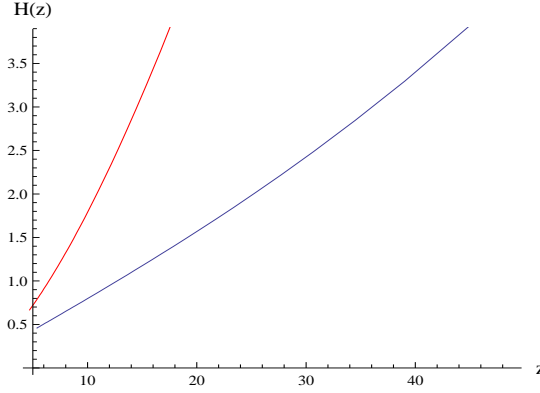


Figure 5: Hubble parameter $H(z)$ versus the Redshift parameter z for the present model (blue) again shows a wide deviation with the standard model $a \propto t^{\frac{2}{3}}$ (red), while blue now is below the red. The plot is for $a_4 = 1$.

8 Summary

We have made a detailed analysis corresponding to the cosmological evolution with $F(R) = R^{\frac{3}{2}}$ theory of gravity, in the Robertson-Walker minisuperspace model. We enlist the present findings.

1. Noether symmetry of $F(R)$ theory of gravity does not admit anything else than $F(R) = R^{\frac{3}{2}}$ in the Robertson-Walker minisuperspace model.
2. The speciality of such a curvature invariant term is apparent under a change of variable from the scale factor a to the basic variable $h_{ij} = a^2$, since h_{ij} becomes cyclic. Thus we conclude that Noether symmetry is in-built in $R^{\frac{3}{2}}$ theory of gravity, for Robertson-Walker minisuperspace model. Under the change of variable, the field equation looks pretty simple and is solved at once.
3. If one tries to find Noether symmetry of $F(R)$ theory of gravity, using the basic variable h_{ij} , the conserved current is solved directly, and it is no more required to find cyclic co-ordinate.
4. The solution in the radiation era ($a \propto t^{\frac{3}{4}}$), is substantially different from the standard Friedmann solution $a \propto t^{\frac{1}{2}}$, which indicates much faster (1.5 times) expansion rate of the Universe. This clearly puts up severe problem in Large scale structure formation (LSS).
5. The solution in the early matter dominated era tracks $t^{\frac{1}{2}}$ power law, rather than standard $t^{\frac{2}{3}}$, like any R^{-n} , with $n > 0$ model, which again creates problem in structure formation and fitting WMAP data, as discussed earlier [8].

6. The parametric values which fit SNIa data, pushes the redshift at decoupling (z_{dec}), which corresponds to 3.75×10^{-4} Billion years in the standard model, to very early epoch ($z_{dec} \approx 200$). However, if this data is fixed a-priori, then transition redshift (deceleration to acceleration) occurs at unacceptably large red-shift.

Thus, unless supplemented by Einstein-Hilbert term, $R^{\frac{3}{2}}$ term alone suffers from serious disease. Since, Noether symmetry of $F(R)$ theory of gravity does not admit anything other than $R^{\frac{3}{2}}$, in the Robertson-Walker minisuperspace model. So, either we have to abandon Noether symmetry or try to involve more degrees of freedom in the theory, by incorporating either scalar field or working with some anisotropic minisuperspace model. This we pose in a future communication.

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